ALGEBRAIC GEOMETRY - MID SEMESTER EXAM

Time: 3 hours   Instructor: Amit Tripathi

Each question is worth 10 marks. Maximum marks: 100

All our usual conventions apply e.g. k is a field, R is a commutative ring with 1 etc. Assume that all rings are noetherian. Unless stated otherwise, you may assume that the field is algebraically closed.

Question 1 (a). Let I_1, $I_2 \subset R$ be ideals. Suppose $I_1 + I_2 = 1$ then $I_1 I_2 = I_1 \cap I_2$. Now without this assumption show that $\sqrt{I_1 I_2} = \sqrt{I_1 \cap I_2}$.

Question 1 (b). Let $I \subset R$ be any ideal. Suppose there exists a prime ideal p such that $I^k \subset p \subset I$ for some $k \in \mathbb{Z}^+$. Show that $p = I$.

Question 2. Find all the automorphisms of $\mathbb{A}^1$.

Question 3. Let I_1, I_2 be ideals in the polynomial ring $k[x_1, \ldots x_n]$. Show that if I_2 is not contained in any of the associated primes of I_1 then $(I_1 : I_2) = I_1$.

Question 4. Let $I \subset R$ be any ideal and $a \in R$. Suppose that for some integer $M \geq 0$ we have $(I : a^M) = (I : a^{M+1})$. Show that $\cup_m (I : a^m) = (I : a^M)$.

Question 5. [**Spectrum of a ring**] Let R be a commutative ring with 1. Consider the set $\mathrm{Spec}(R) := \{p \subset R |\ p \text{ is a prime ideal}\}$. Describe $\mathrm{Spec}\, \mathbb{C}[x]$ and $\mathrm{Spec}\, \mathbb{R}[x]$.

Question 6. Describe a rational map $\mathbb{A}^1 \to C$ where $C \subset \mathbb{A}^2$ is the curve given by $V(y^2 - x^3)$.

Question 7. Don't assume k is algebraically closed for this problem. Let $\mathfrak{a} = (xy, yz, xz) \subset k[x, y, z]$. Is $\mathfrak{a} = I(V(\mathfrak{a}))$? Prove that $\mathfrak{a}$ can't be generated by 2 elements.

Question 8. Don't assume k is algebraically closed for this problem. Let $\mathfrak{a} = (x^2 + y^2 - 1, y - 1)$. Describe $I(V(\mathfrak{a})) \setminus \mathfrak{a}$.

Question 9. Let $X = V(x_1 x_4 - x_2 x_3) \subset \mathbb{A}^4$. Show that $A(X)$ is not a UFD? Find a height 1 prime which is not principal. If $\varphi : \mathbb{A}^4 \to \mathbb{A}^1$ be a rational morphism defined as $\varphi = x_1/x_2$ then find the domain of φ.

Question 10. Let $\varphi : \mathbb{A}^1 \to \mathbb{A}^n$ be given as $t \to (t, t^2, t^3, \ldots t^n)$. Show that the image of φ is an affine variety and show that φ describes an isomorphism (of affine varieties) of $\mathbb{A}^1$ with the image.